Digital Communication Systems
ECS 452

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4. *Mutual Information* and Channel Capacity

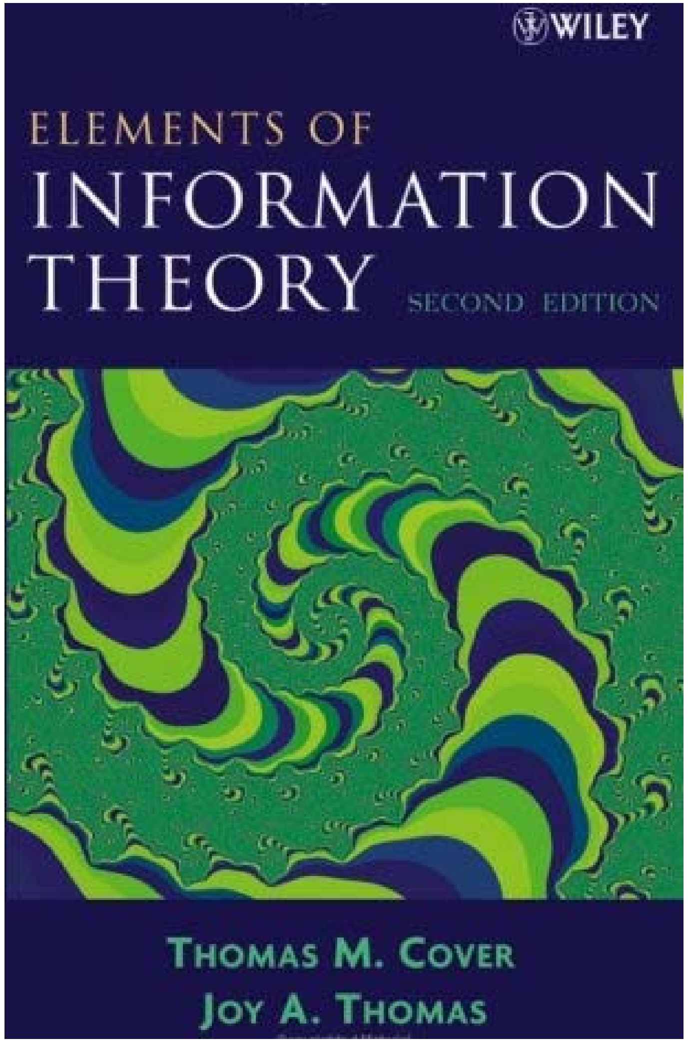
Office Hours:

BKD, 6th floor of Sirindhralai building

Monday	10:00-10:40
Tuesday	12:00-12:40
Thursday	14:20-15:30

Reference for this chapter

- Elements of Information Theory
- By Thomas M. **Cover** and Joy A. **Thomas**
- 2nd Edition (Wiley)
- Chapters 2, 7, and 8
- 1st Edition available at SIIT library: Q360 C68 1991



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Information-Theoretic Quantities

Conditional Entropies

Amount of randomness in Y

$$H(Y) \equiv -\sum_{y\in\mathcal{Y}} q(y)\log_2 \overbrace{q(y)}^{P[Y=y]} \equiv H(\underline{\mathbf{q}})$$

Amount of randomness still remained in Y when we know that $X = x$.

$$\overbrace{H(Y|X=x)}^{\text{given a particular value } x} \equiv H(Y|x) \equiv -\sum_{y\in\mathcal{Y}} Q(y|x)\log_2 \overbrace{Q(y|x)}^{P[Y=y|X=x]}$$

Apply the entropy calculation to a row from the $\mathbf{Q}$ matrix

The **average** amount of randomness still remained in Y when we know X

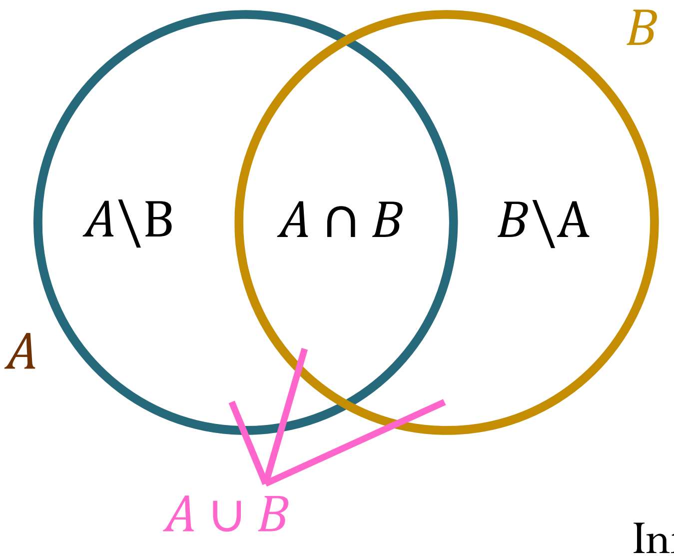
$$H(Y|X) \equiv \sum_{x\in\mathcal{X}} p(x)H(Y|x)$$

$$= H(X,Y) - H(X)$$

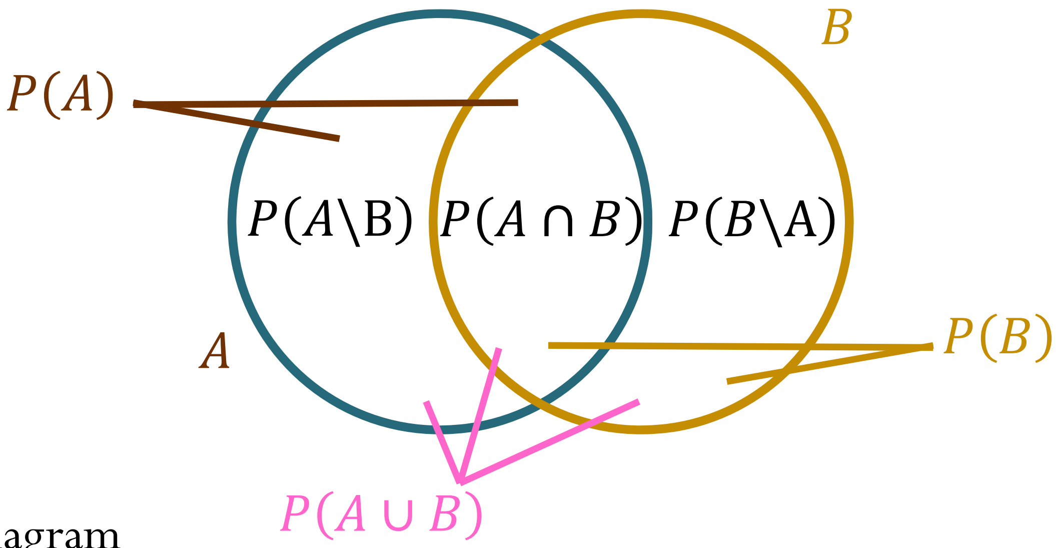
$$= H(Y) - I(X;Y)$$

Diagrams [Figure 16]

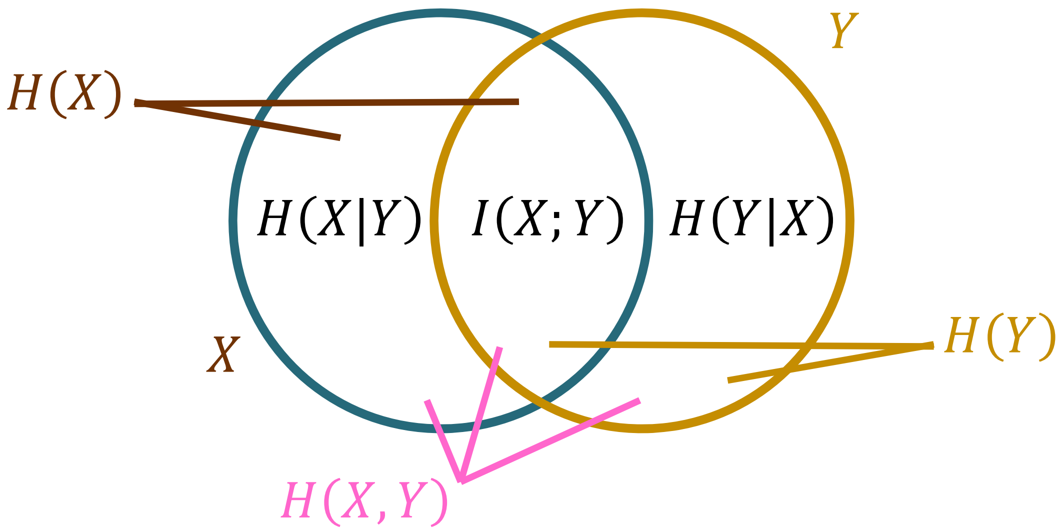
Venn Diagram



Probability Diagram

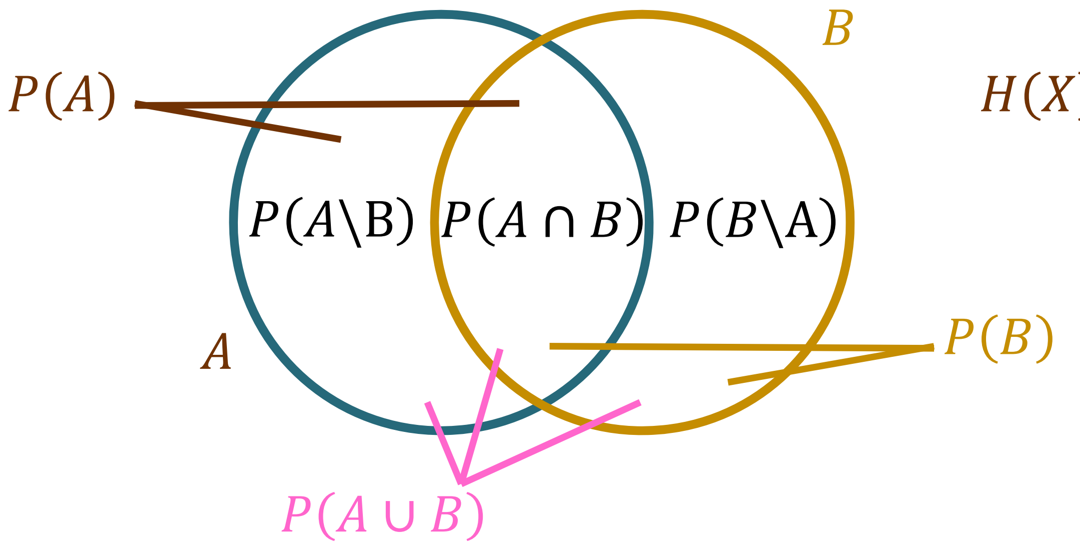


Information Diagram

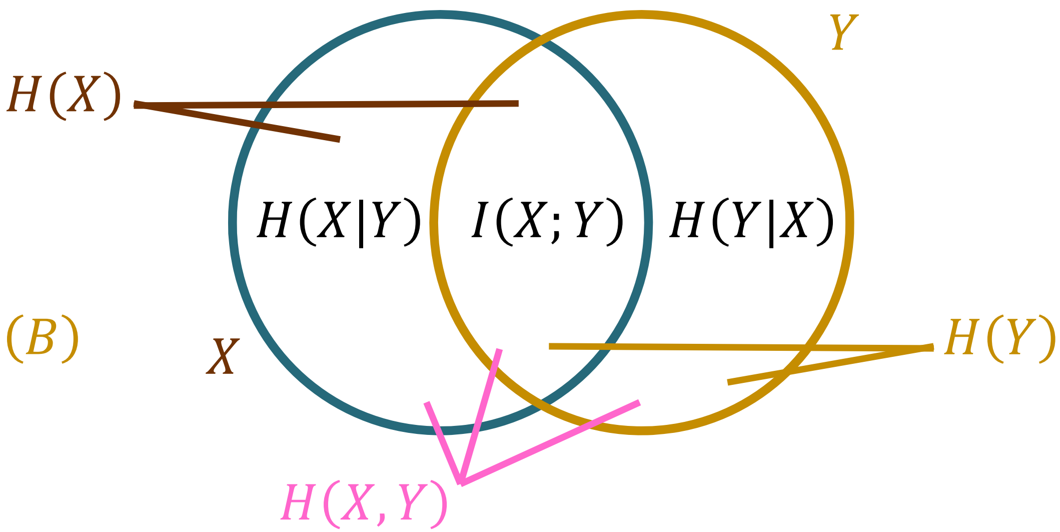


Diagrams [Figure 16]

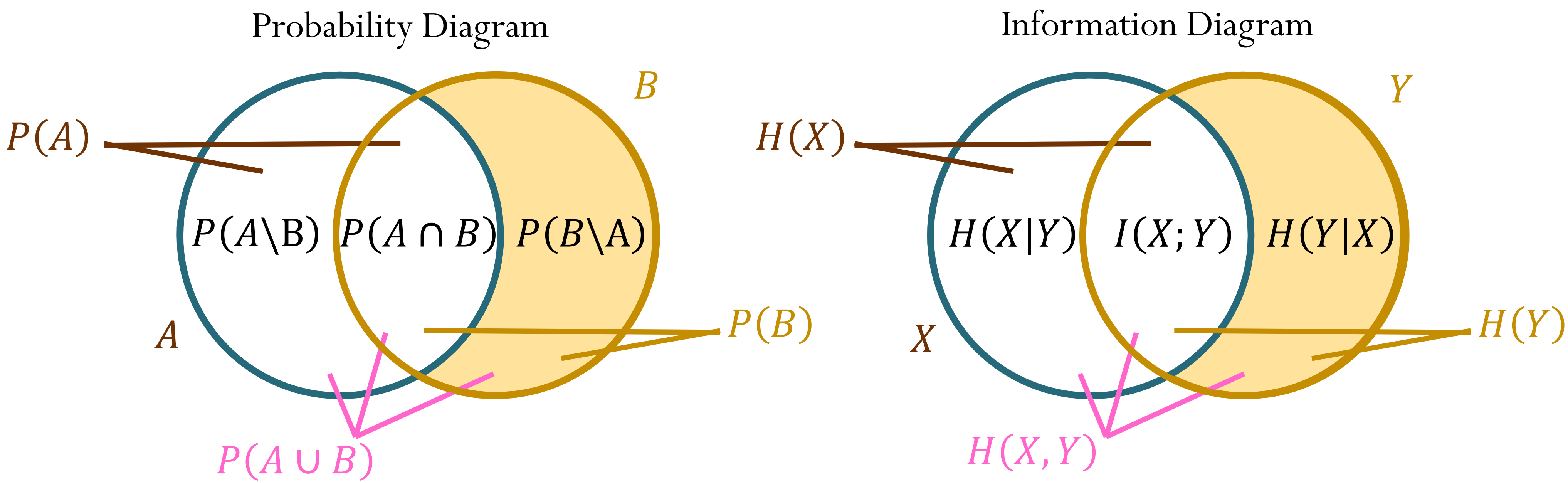
Probability Diagram



Information Diagram



Diagrams



$$P(B \setminus A) = P(A \cup B) - P(A) \qquad H(Y|X) = H(X,Y) - H(X)$$

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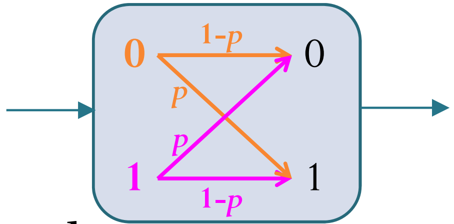
Operational Channel Capacity

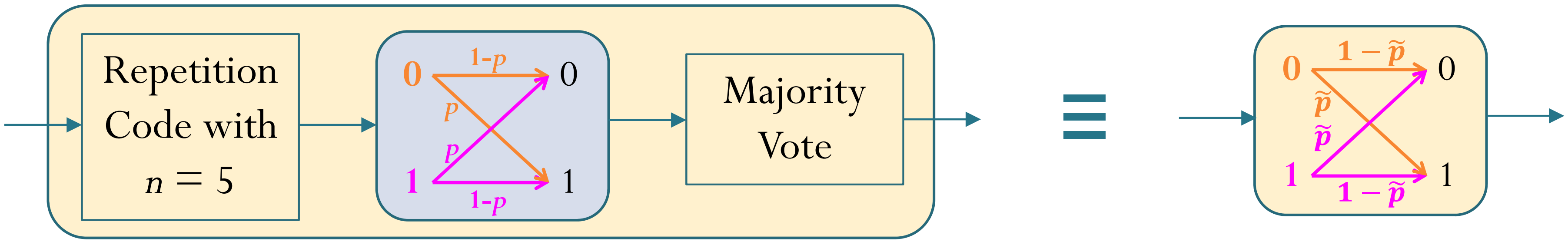
Example: Repetition Code

[Figure 14]

- Original Equivalent Channel:



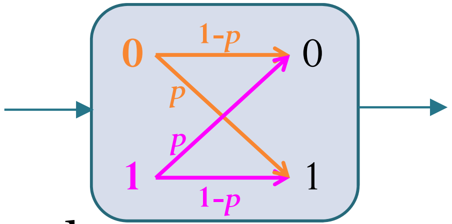
  - BSC with crossover probability $p = 0.01$
- New (and Better) Equivalent Channel:



  - Use repetition code with $n = 5$ at the transmitter
  - Use majority vote at the receiver
  - New BSC with $\tilde{p} = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5(1-p)^0 \approx 10^{-5}$

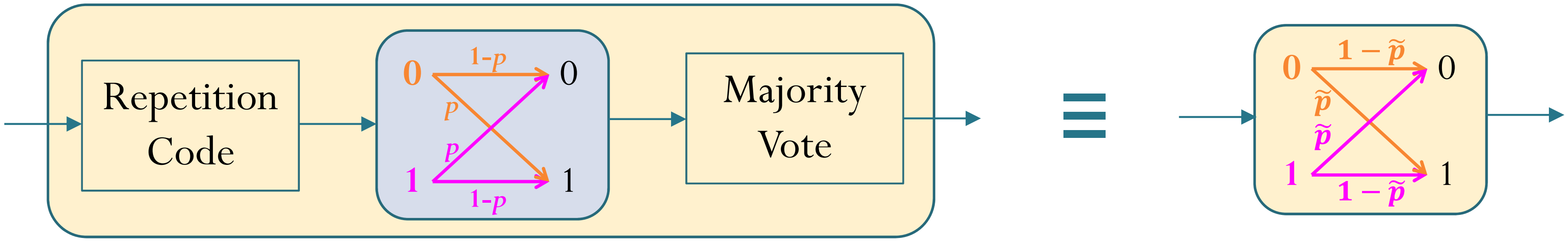
Example: Repetition Code

- Original Equivalent Channel:



  - BSC with crossover probability $p = 0.1$

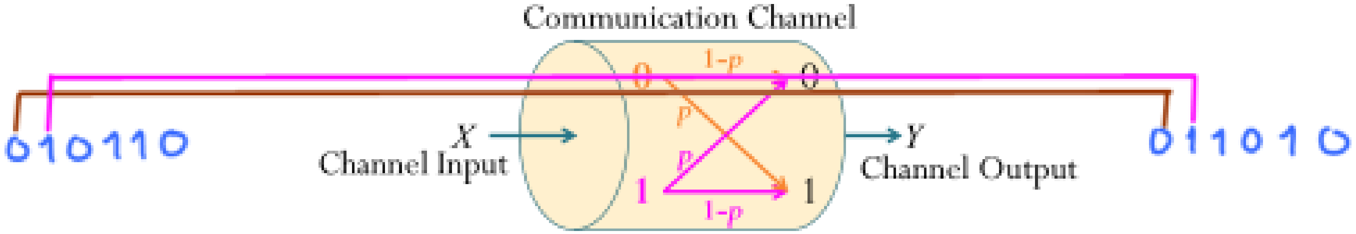
- New (and Better) Equivalent Channel:



  - Use repetition code at the transmitter
  - Use majority vote at the receiver
  - New BSC with new crossover probability $\tilde{p}$

Reminder [From ECS315]

Example 6.58. ***Digital communication over unreliable channels***: Consider a digital communication system through the **binary symmetric channel** (BSC) discussed in Example 6.18. We repeat its compact description here.



Suppose you put 010110 into this channel. What is the probability that you get 011010 as output?

$(1-p)(1-p)pp(1-p)(1-p)$

$= (1-p)^4 p^2$

85

$P(\mathcal{E}) = P(\mathcal{E} | [X=0]) P[X=0] + P(\mathcal{E} | [X=1]) P[X=1]$

$= p \times P[X=0] + p\, P[X=1] = p\,(\underbrace{P[X=0] + P[X=1]}_{1}) = p$

Again this channel can be described as a channel that introduces

One method of reducing the error rate is to use error-correcting codes:

[From ECS315]

A simple error-correcting code is the ***repetition code***. Exam-

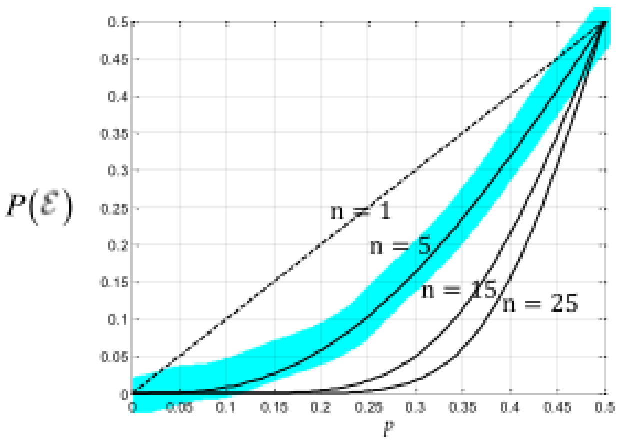
Two ways to calculate the probability of error:

(a) (transmission) error occurs if and only if the number of bits in error are ≥ 3.

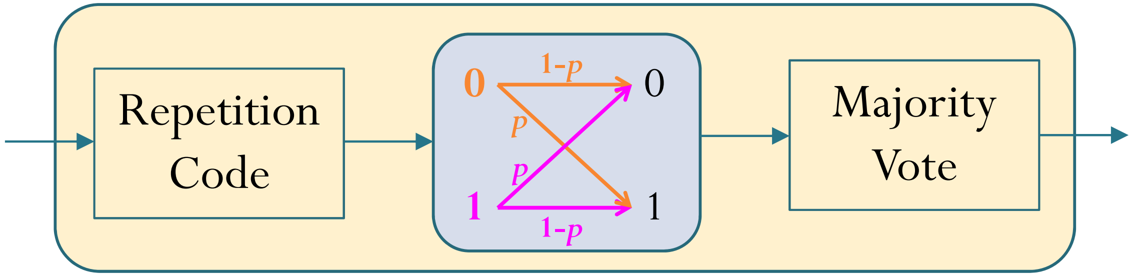
$$\left(P[\hat{M} \neq M] = \right) P(\mathcal{E}) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + \binom{5}{5}p^5(1-p)^0$$

(b) (transmission) error occurs if and only if the number of bits *not* in error are ≤ 2.

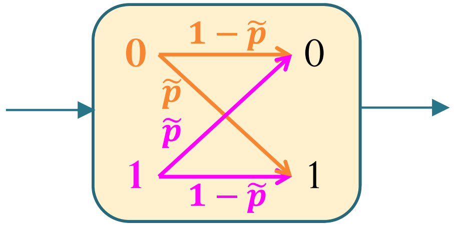
$$P(\mathcal{E}) = \binom{5}{0}(1-p)^0p^5 + \binom{5}{1}(1-p)^1p^4 + \binom{5}{2}(1-p)^2p^3$$



Example: Repetition Code



≡



n	$\tilde{p}$
1	$p = 0.1$
3	$\binom{3}{2}p^2(1-p)+\binom{3}{3}p^3 \approx 0.0280$
5	$\binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)^1+\binom{5}{5}p^5 \approx 0.0086$
7	≈ 0.0027
9	$\approx 8.9092\times10^{-4}$
11	$\approx 2.9571\times10^{-4}$

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Information Channel Capacity

Channel Capacity

Channel Capacity

"**Operational**": max rate at which **reliable** communication is possible

Arbitrarily small error probability can be achieved.

"**Information**": $\max_{\underline{\mathbf{p}}} I(X;Y)$ [bpcu]

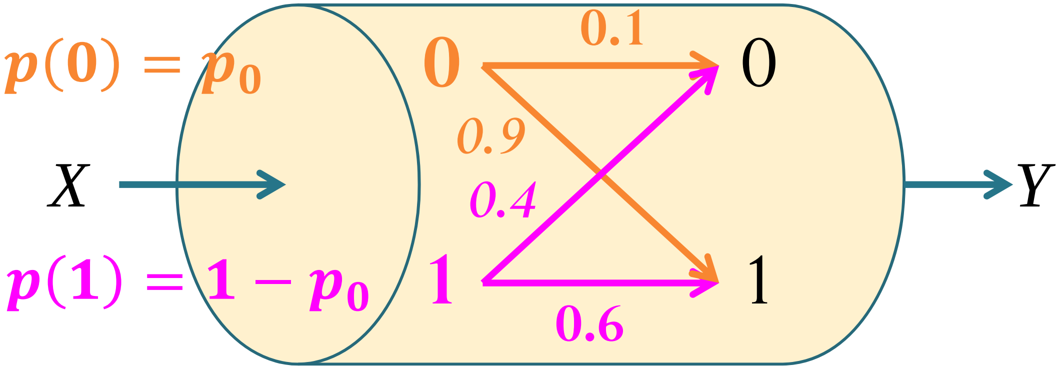
Shannon [1948] shows that these two quantities are actually the same.

MATLAB

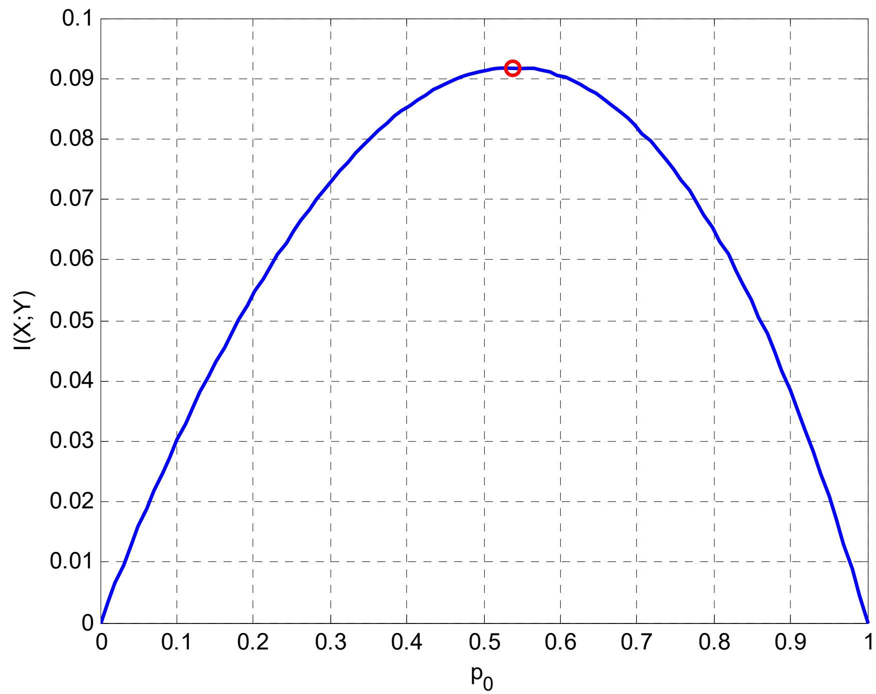
```
function H = entropy2s(p)
% ENTROPY2 accepts probability mass function
% as a row vector, calculate the corresponding
% entropy in bits.
p=p(find(abs(sort(p)-1)>1e-8)); % Eliminate 1
p=p(find(abs(p)>1e-8)); % Eliminate 0
if length(p)==0
    H = 0;
else
    H = simplify(-sum(p.*log(p))/log(sym(2)));
end
```

```
function I = informations(p,Q)
X = length(p);
q = p*Q;
HY = entropy2s(q);
temp = [];
for i = 1:X
    temp = [temp entropy2s(Q(i,:))];
end
HYgX = sum(p.*temp);
I = HY-HYgX;
```

Capacity calculation for BAC

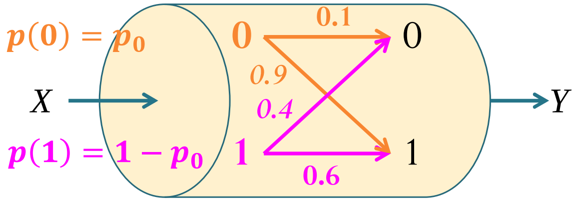


$$Q = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}$$



Capacity of 0.0918 bits is achieved by $\underline{p} = [0.5380, \ 0.4620]$

Capacity calculation for BAC

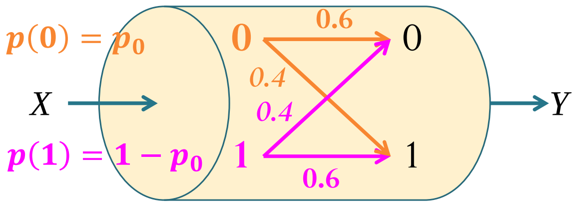


$$Q = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}$$

```
close all; clear all;
syms p0
p = [p0 1-p0];
Q = [1 9; 4 6]/sym(10);

I = simplify(informations(p,Q))

p0o = simplify(solve(diff(I)==0))

po = eval([p0o 1-p0o])

C = simplify(subs(I,p0,p0o))

eval(C)
```

```
>> Capacity_Ex_BAC
I =
(log(2/5 - (3*p0)/10)*((3*p0)/10 - 2/5) - log((3*p0)/10 + 3/5)*((3*p0)/10 + 3/5))/log(2) + (log((5*2^(3/5)*3^(2/5))/6)*(p0 - 1))/log(2) + (p0*log((3*3^(4/5))/10))/log(2)
p0o =
(27648*2^(1/3))/109565 - (69984*2^(2/3))/109565 + 135164/109565
po =
    0.5376    0.4624
C =
(log((3*3^(4/5))/10)*((27648*2^(1/3))/109565 - (69984*2^(2/3))/109565 + 135164/109565))/log(2) - (log((104976*2^(2/3))/547825 - (41472*2^(1/3))/547825 + 16384/547825)*((104976*2^(2/3))/547825 - (41472*2^(1/3))/547825 + 16384/547825) + log((41472*2^(1/3))/547825 - (104976*2^(2/3))/547825 + 531441/547825)*((41472*2^(1/3))/547825 - (104976*2^(2/3))/547825 + 531441/547825))/log(2) + (log((5*2^(3/5)*3^(2/5))/6)*((27648*2^(1/3))/109565 - (69984*2^(2/3))/109565 + 25599/109565))/log(2)
ans =
    0.0918
```

Same procedure applied to BSC



$$Q = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

```
close all; clear all;
syms p0
p = [p0 1-p0];
Q = [6 4; 4 6]/sym(10);

I = simplify(informations(p,Q))

p0o = simplify(solve(diff(I)==0))

po = eval([p0o 1-p0o])

C = simplify(subs(I,p0,p0o))

eval(C)
```

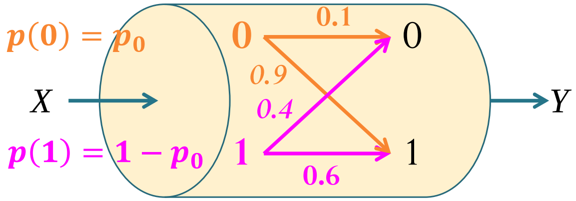
```
>> Capacity_Ex_BSC
I =
(log((5*2^(3/5)*3^(2/5))/6)*(p0 - 1))/log(2) -
(p0*log((5*2^(3/5)*3^(2/5))/6))/log(2) - (log(p0/5 +
2/5)*(p0/5 + 2/5) - log(3/5 - p0/5)*(p0/5 -
3/5))/log(2)
p0o =
1/2
po =
   0.5000   0.5000
C =
log((2*2^(2/5)*3^(3/5))/5)/log(2)
ans =
   0.0290
```

Blahut–Arimoto algorithm

```
function [ps C] = capacity_blahut(Q)
% Input:      Q  = channel transition probability matrix
% Output:     C  = channel capacity
%             ps = row vector containing pmf that achieves capacity

tl = 1e-8; % tolerance (for the stopping condition)
n = 1000; % max number of iterations (in case the stopping condition
          % is "never" reached")
nx = size(Q,1); pT = ones(1,nx)/nx; % First, guess uniform X.
for k = 1:n
    qT = pT*Q;
    % Eliminate the case with 0
    % Column-division by qT
    temp = Q.*(ones(nx,1)*(1./qT));
    %Eliminate the case of 0/0
    l2 = log2(temp);
    l2(find(isnan(l2) | (l2==-inf) | (l2==inf)))=0;
    logc = (sum(Q.*(l2),2))';
    CT = 2.^(logc);
    A = log2(sum(pT.*CT)); B = log2(max(CT));
    if((B-A)<tl)
        break
    end
    % For the next loop
    pT = pT.*CT;     % un-normalized
    pT = pT/sum(pT); % normalized
    if(k == n)
        fprintf('\nNot converge within n loops\n')
    end
end
ps = pT;
C = (A+B)/2;
```

[capacity_blahut.m]

Capacity calculation for BAC: a revisit



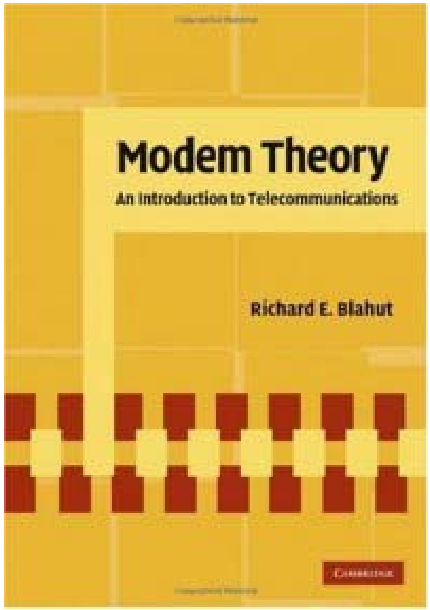
$$Q = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}$$

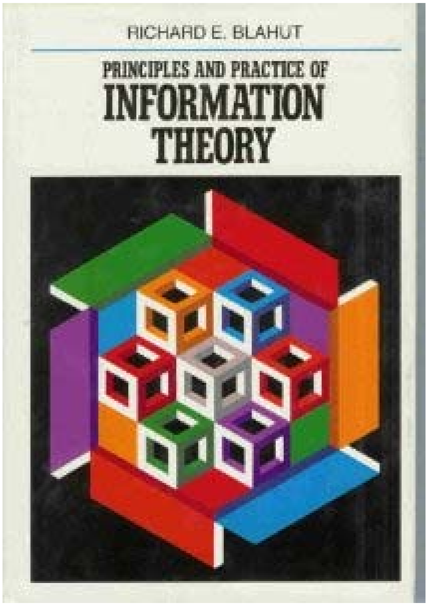
```
close all; clear all;

Q = [1 9; 4 6]/10;

[ps C] = capacity_blahut(Q)
```

```
>> Capacity_Ex_BAC_blahut
ps =
   0.5376   0.4624
C =
   0.0918
```

Richard Blahut

- Former chair of the Electrical and Computer Engineering Department at the University of Illinois at Urbana-Champaign
- Best known for **Blahut–Arimoto algorithm (Iterative Calculation of C)**





Claude E. Shannon Award

Claude E. **Shannon** (1972)
David S. Slepian (1974)
Robert M. Fano (1976)
Peter Elias (1977)
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Jacob Wolfowitz (1979)
W. Wesley Peterson (1981)
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[http://www.itsoc.org/honors/claude-e-shannon-award]

Berger plaque

Raymond Yeung

- BS, MEng and PhD degrees in electrical engineering from **Cornell** University in 1984, 1985, and 1988, respectively.